Exercice 6 :

1) a- A_i : "le i-ème jeton tiré porte le numéro 1"

$$\{X=k\} = (A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}^c) \uplus (A_1^c \cap A_2^c \cap \ldots \cap A_k^c \cap A_{k+1})$$

$$P(X=k) = P(A_1 \cap \ldots \cap A_k \cap A_{k+1}^c) + P(A_1^c \cap \ldots \cap A_k^c \cap A_{k+1})$$

$$= p^k(1-p) + (1-p)^k p$$

$$\sum_{k=1}^{+\infty} P(X=k) = \sum_{k=1}^{+\infty} p^k(1-p) + (1-p)^k p = (1-p) \times \frac{p}{1-p} + p \times \frac{1-p}{p} = 1 \quad \underline{\text{OK}}$$

$$P(X=k) = p^k(1-p) + (1-p)^k p$$

b- $$E(X) = \sum_{k=1}^{+\infty} k\,P(X=k) = (1-p)\sum_{k=1}^{\infty} k p^k + p \sum_{k=1}^{\infty} k(1-p)^k$$

$$= (1-p) \times p \sum_{k=1}^{\infty} k p^{k-1} + p(1-p) \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p} \Rightarrow \sum_{k=1}^{\infty} k p^{k-1} = \left(\frac{1}{1-p}\right)' = \frac{1}{(1-p)^2}$$

$$\sum_{k=0}^{\infty} (1-p)^k = \frac{1}{p} \Rightarrow -\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{-1}{p^2}$$

$$E(X) = (1-p) \times p \times \frac{1}{(1-p)^2} + p(1-p) \times \frac{1}{p^2}$$

$$E(X) = \frac{p}{1-p} + \frac{(1-p)}{p}$$

2) Montrons que $\forall (i,j) \in \mathbb{N}^{*2}$, $P(X=i, Y=j) = p^{i+1} q^j + q^{i+1} p^j$

$$\{X=i\} \cap \{Y=j\} = \left(A_1 \cap A_2 \cap \dots \cap A_i \cap A_{i+1}^c \cap A_{i+2}^c \cap \dots \cap A_{i+j}^c\right)$$

$$\uplus \left(A_1^c \cap A_2^c \cap \dots \cap A_i^c \cap A_{i+1} \cap \dots \cap A_{i+j}\right)$$

$$\Rightarrow P(X=i, Y=j) = P\left(\bigcap_{k=1}^{i} A_k \cap A_{k+j}^c\right) + P\left(\bigcap_{k=1}^{i} A_k^c \cap A_{k+j}\right)$$

$$= p^i (1-p)^j + (1-p)^i p^j$$

3) a- Montrons que $P(X=Y) = \dfrac{pq}{1-pq}$

$$P(X=Y) = \sum_{k=1}^{\infty} P(X=k, Y=k) = \sum_{k=1}^{\infty} p^k (1-p)^k + (1-p)^k q^k$$

$$= 2 \sum_{k=1}^{\infty} p^k (1-p)^k = 2 \sum_{k=1}^{\infty} (pq)^k$$

$$= 2 \frac{pq}{1-pq}$$