Exercice 2

Pour tout k *de* $\mathbb{N}^*$, *on pose* $v_k = \ln\left(1 + \dfrac{2}{k(k+3)}\right)$.

1) *En factorisant* $k^2 + 3k + 2$, *écrire* v_k *sous la forme* $u_{k+1} - u_k$.
2) *En déduire que la série de terme général* v_n *converge et donner la valeur de* $\sum_{n=1}^{+\infty} v_n$.

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1) Factorisons k^2+3k+2.

$\Delta = 9-4\times1\times2 = 9-8 = 1$

$x_1 = \frac{-3+1}{2} = \frac{-2}{2} = -1$

$x_2 = \frac{-3-1}{2} = -\frac{4}{2} = -2$.

On a donc : $\boxed{k^2+3k+2 = (k+1)(k+2)}$

Ainsi, on a $v_k = \ln\left(1+\frac{2}{k(k+3)}\right) = \ln\left(\frac{k(k+3)+2}{k(k+3)}\right)$

$= \ln\left(\frac{k^2+3k+2}{k(k+3)}\right) = \ln\left(\frac{(k+1)(k+2)}{k(k+3)}\right) = \ln\left(\frac{\frac{k+1}{k+3}}{\frac{k}{k+2}}\right)$

$= \ln\left(\frac{k+1}{k+3}\right) - \ln\left(\frac{k}{k+2}\right)$.

En posant $U_k = \ln\left(\frac{k}{k+2}\right)$, on a donc :

$$\boxed{V_k = U_{k+1} - U_k = \ln\left(\frac{k+1}{k+3}\right) - \ln\left(\frac{k}{k+2}\right)}$$

2) $\sum_{n=1}^{+\infty} V_n = \sum_{n=1}^{+\infty} U_{n+1} - U_n = \sum_{n=1}^{+\infty} \ln\left(\frac{n+1}{n+3}\right) - \ln\left(\frac{n}{n+2}\right)$

$= U_2 - U_1 + U_3 - U_2 \ldots + U_{n+1} - U_n$. Par télescopage, on obtient :

$\sum_{n=1}^{+\infty} V_n = U_{n+1} - U_1 = \ln\left(\frac{n+1}{n+3}\right) - \ln\left(\frac{1}{3}\right)$. Or, $\lim_{n\to+\infty} \ln\left(\frac{n+1}{n+3}\right) = 0$

On a donc $\sum_{n=1}^{+\infty} V_n = -\ln\left(\frac{1}{3}\right) = -(\ln(1) - \ln(3)) = \boxed{\ln(3)}$